International Journal of Theoretical Physics, Vol. 34, No. 10, 1995

Two-Field Coset Space Realization of the Spin-4 Current Algebra

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Received September 28, 1994

A two-field realization of the spin-4 current algebra is obtained through the coset space approach. The transformation properties of these two fields are also deduced through the composition law, which turns out to be a simple generalization of the corresponding diffeomorphic transformation.

1. INTRODUCTION

Nonlinear realizations of compact semisimple Lie groups were first studied many years ago, while investigating the structure of phenomenological Lagrangians (Coleman *et al.*, 1969; Callan *et al.*, 1969). Recently, the idea of nonlinear realizations has been revived: this time it has been applied to infinite-dimensional symmetries. These include the Virasoro, W_3 (Ivanov *et al.*, 1992a), super-W_3 (Ivanov *et al.*, 1992b), and $W_{1+\infty}$ (Stelle and Sezgin, 1992; Belluci *et al.*, 1992) algebras. This has reinforced further the well-known relation between these algebras and integrable systems. For example, the W_3 algebra is the second Hamiltonian structure of the Boussinesq equation. Conversely, the Boussinesq equation has been derived using the coset realization of W_3. The technique has also allowed the construction of two-field and multifield coset space realizations of $W_{1+\infty}$ (Stelle and Sezgin, 1992; Belluci *et al.*, 1992).

The spin-4 current algebra was first written down some years ago (Hamada and Takao, 1988). This algebra has a spin-4 current besides the conformal energy-momentum tensor. Higher spin currents e.g., spin 6, arise as normal ordered products of lower spin fields. In fact, in this way the

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0020-7748/95/1000-2123$07.50/0

nonlinear spin-4 current algebra has been linearized. In this paper, we shall construct a two-field realization of the spin-4 current algebra using the coset realization method. The basic idea is to choose a subalgebra of the given infinite-dimensional Lie algebra. After constructing the coset, the Cartan form is written down explicitly. The action of the group on the coset then determines the transformations of the fields. These are functions of the fields and coordinate variations. Finally, imposition of covariant inverse Higgs-type constraints allows one to express all fields in terms of only two. So a two-field realization of the above algebra is obtained.

2. THE SPIN-4 CURRENT ALGEBRA

The spin-4 current algebra is described by the following commutation relations:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12} cm(m^2 - 1)\delta_{m+n,0} \tag{1}$$

$$[L_m, V_n] = (3m - n)V_{m+n} \tag{2}$$

$$\begin{aligned}
[V_m, V_n] = {} & \frac{C}{(4 \times 7!)} m(m^2 - 1)(m^2 - 4)(m^2 - 9)\delta_{m+n,0} \\
& + (m - n)\left\{\frac{1}{1680} (3m^4 - 2m^3n + 4m^2n^2 - 2mn^3 \right. \\
& + 3n^4 - 39m^2 + 20mn \\
& - 39n^2 + 108)L_{m+n} + \frac{1}{28(5c + 22)} (39m^2 - 20mn + 39n^2 \\
& + 57m + 57n - 102)\Lambda_{m+n} \\
& + \frac{1}{(2c - 1)(7c + 68)} \left[-\frac{3}{20} (19c - 524)\Xi_{m+n} \right. \\
& \left.\left. + \frac{12(72c + 13)}{5c + 22} \Delta_{m+n}\right]\right\}
\end{aligned} \tag{3}$$

where Λ_k, Δ_k, Ξ_k are higher spin operators (spin 4, spin 6, spin 6, respectively). V_k is a spin-4 operator. Our interest will be confined to a subalgebra of the above spin-4 current algebra. This subalgebra is generated by $\{L_{-1}, L_0, L_1, \ldots, V_{-3}, V_{-2}, V_{-1}, \ldots, \Lambda_{-3}, \Lambda_{-2}, \ldots, \Delta_{-5}, \Delta_{-4}, \ldots, \Xi_{-5}, \Xi_{-4}, \ldots\}$ and is a generalization of the truncated Virasoro algebra considered in Ivanov *et al.* (1992a).

3. NONLINEAR REALIZATION OF THE SPIN-4 CURRENT ALGEBRA

We shall begin by choosing a subalgebra of the 'truncated' spin-4 current algebra. One possible subalgebra is

$$\{L_0, L_{-1} + \sqrt{70}\, V_{-1}, V_{-3}, V_{-2}, V_0, \Lambda_{-3}, \Lambda_{-2}, \ldots,$$
$$\Delta_{-5}, \Delta_{-4}, \ldots, \Xi_{-5}, \Xi_{-4}, \ldots\} \tag{4}$$

The generators left out of the subalgebra, i.e., those belonging to the coset, are L_{-1}, L_1, L_2, ..., V_1, V_2, ... As usual, it is appropriate to associate a coordinate x with L_{-1}. Fields are associated with the other generators. It will turn out that all fields can be expressed in terms of only two by using the inverse Higgs effect.

An element of the coset space can be written as

$$g = e^{xL_{-1}} \prod_{n\geq 3} e^{\Psi_n L_n} \prod_{n\geq 5} e^{\xi_n V_n} e^{uL_2} e^{wV_4} e^{u_1 L_1} e^{w_1 V_1} e^{w_2 V_2} e^{w_3 V_3} \tag{5}$$

The Cartan form is

$$\Omega = g^{-1}\, dg = \sum_{n\geq -1} \omega_n L_n + \sum_{n\geq -3} \alpha_n V_n \tag{6}$$

The action of the group on the coset is defined as follows:

$$g_0 \cdot g(x, u_1, u, \Psi_n, w_1, w_2, w_3, w, \xi_n)$$
$$= g(\tilde{x}, \tilde{u}_1, \tilde{u}, \tilde{\Psi}_n, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}, \tilde{\xi}_n) \cdot h(x, g_0) \tag{7}$$

where g_0 is an arbitrary element of the group and h is an induced transformation from the stability subgroup corresponding to (4). We write down some of the ω_n's and α_n's:

$$\omega_{-1} = dx$$
$$\omega_0 = -2u_1\, dx$$
$$\omega_1 = \left(-3u + u_1^2 + \frac{3}{35} w_1^2\right) dx + du_1$$
$$\omega_2 = -4\Psi_3\, dx + du$$
$$\omega_3 = \left(-5\Psi_4 + \frac{3}{2} u^2 - 4u_1\Psi_3\right) dx + d\Psi_3 + u_1\, du$$
$$\omega_4 = -\left\{6\Psi_5 + 4u_1^2\Psi_3 - 2u_1\left(-5\Psi_4 + \frac{3}{2} u^2\right)\right\} dx$$
$$+ 2u_1\, d\Psi_3 + u_1^2\, du + d\Psi_4 \tag{8}$$

$$
\begin{aligned}
\alpha_n &= 0 \qquad \text{for} \quad -3 \leq n \leq -1 \\
\alpha_0 &= -4w_1\, dx \\
\alpha_1 &= (2w_1u_1 - 5w_2)\, dx + dw_1 \\
\alpha_2 &= \left\{4w_2u_1 + 2w_1(-3u + u_1^2) - 6w_3 + \frac{2}{35} w_1^3\right\} dx \\
&\quad + 2w_1\, du_1 + dw_2 \\
\alpha_3 &= \left\{-7w - 20w_1\Psi_3 + w_2\left(-3u + u_1^2 + \frac{3}{35} w_1^2\right) + 6w_3u_1\right\} dx \\
&\quad + 5w_1\, du + w_2\, du_1 + dw_3 \qquad (9) \\
\alpha_4 &= 8\left\{-\xi_5 + w_1\left(-5\Psi_4 + \frac{3}{2} u^2 - 4u_1\Psi_3 - 2w_2\Psi_3\right\} dx \\
&\quad + 8w_1 d\Psi_3 + (8w_1u_1 + 4w_2)\, du + dw
\end{aligned}
$$

To determine the transformations of the fields under (7) one has to use the equation

$$\Omega = h^{-1}\tilde{\Omega}h + h^{-1}\, dh \qquad (10)$$

where

$$\tilde{\Omega} = g^{-1}(\tilde{x}, \tilde{u}_1, \ldots)\, dg(\tilde{x}, \tilde{u}_1, \ldots) \qquad (11)$$

In our case

$$h = e^{a_0L_0}e^{\mu V_{-3}}e^{b_{-2}V_{-2}}e^{b_{-1}V_{-1}}e^{b_0V_0}\tilde{h} \qquad (12)$$

Here $\tilde{h}$ stands for the factors spanned by the higher spin (spin 6) generators and the spin-4 generator Λ_k. The parameters a_0, μ, b_{-2}, b_{-1}, b_0 are all infinitesimal. Then we have the following equations from (10):

$$0 = -b_{-2}\tilde{\omega}_{-1} + 3\mu\tilde{\omega}_0 + \mathbf{d}\mu \qquad (13)$$

$$0 = -2b_{-1}\tilde{\omega}_{-1} + 2b_{-2}\tilde{\omega}_0 + 6\mu\tilde{\omega}_1 + db_{-2} \qquad (14)$$

$$0 = -3b_0\tilde{\omega}_{-1} + b_{-1}\tilde{\omega}_0 + 5b_{-2}\tilde{\omega}_1 + 9\mu\omega_2 + db_{-1} \qquad (15)$$

$$\alpha_0 = 4b_{-1}\tilde{\omega}_1 + 8b_{-2}\tilde{\omega}_2 + 12\mu\tilde{\omega}_3 + \tilde{\alpha}_0 + db_0 \qquad (16)$$

$$\omega_{-1} = \tilde{\omega}_{-1} - a_0\tilde{\omega}_{-1} + \frac{3}{70} b_{-1}\tilde{\alpha}_0 - \frac{1}{14} b_{-2}\tilde{\alpha}_1 + \frac{3}{14} \mu\tilde{\alpha}_2 \qquad (17)$$

$$\omega_0 = \tilde{\omega}_0 + \frac{1}{35} b_{-1}\tilde{\alpha}_1 - \frac{1}{7} b_{-2}\tilde{\alpha}_2 + \frac{9}{7} \mu\tilde{\alpha}_3 + da_0 \qquad (18)$$

Equations (16) and (18) determine the variations $\delta u_1 = \tilde{u}_1(x) - u_1(x)$ and $\delta w_1 = \tilde{w}_1(x) - w_1(x)$ in terms of the coordinate variation $\lambda(x)$, μ, u_1, w_1, u, Ψ_3, Ψ_4, w_2, w_3, etc.

4. THE INVERSE HIGGS EFFECT

We shall now impose the inverse Higgs constraints on the Cartan forms (8) and (9). We have

$$\begin{aligned} \omega_n &= 0 \qquad \text{for} \quad n \geq 1 \\ \alpha_n &= 0 \qquad \text{for} \quad n \geq 1 \end{aligned} \tag{19}$$

Then we get the following expressions:

$$u = \frac{1}{3}\left(u_1^2 + \frac{3}{35} w_1^2 + u_1'\right) \tag{20}$$

$$\Psi_3 = \frac{1}{12}\left(2u_1u_1' + \frac{6}{35} w_1w_1' + u_1''\right) \tag{21}$$

$$\begin{aligned} \Psi_4 = \frac{1}{5}\Bigg\{&\frac{1}{6} u_1^4 + \frac{1}{6}\left(\frac{3}{35}\right)^2 w_1^4 + \frac{1}{3} u_1'^2 + \frac{1}{35} u_1^2w_1^2 \\ &+ \frac{1}{3} u_1^2u_1' + \frac{1}{35} w_1^2u_1' + \frac{1}{6} u_1u_1'' + \frac{1}{70} w_1w_1'' + \frac{1}{12} u_1'''\Bigg\} \end{aligned} \tag{22}$$

$$w_2 = \frac{1}{5}(2w_1u_1 + w_1') \tag{23}$$

$$w_3 = \frac{4}{15} w_1u_1^2 + \frac{1}{5} w_1'u_1 + \frac{1}{15} w_1u_1' + \frac{1}{30} w_1'' - \frac{2}{105} w_1^3 \tag{24}$$

and so on.

Thus the higher order coset fields are expressed in terms of only two fields: u_1, w_1, as advertised. Using these results in (16) and (18), we get

$$\begin{aligned} \delta u_1 = -u_1\lambda' - \lambda u_1' + \frac{1}{2}\lambda'' - \frac{6}{35}\Bigg\{&\left(\frac{1}{2}\mu''' - 5\mu''u_1 - 8\mu'u_1' - 3\mu u_1''\right. \\ &\left. + 12\mu'u_1^2 + 24\mu u_1u_1'\right)w_1 + \left(\frac{1}{2}\mu'' - 5\mu'u_1\right. \\ &\left. - 3\mu u_1' + 12\mu u_1^2\right)w_1'\Bigg\} \end{aligned} \tag{25}$$

$$\delta w_1 = -\lambda w_1' - w_1\lambda' - \frac{1}{6}\mu_1\left(\frac{1}{2}\mu''' - 5\mu''u_1 - 8\mu'u_1' - 3\mu u_1'' + 12\mu'u_1^2 + 24\mu u_1 u_1'\right) - \frac{1}{6}u_1'\left(\frac{1}{2}\mu'' - 5\mu'u_1 - 3\mu u_1' + 12\mu u_1^2\right) + \frac{1}{12}\left\{\frac{1}{2}b_{-1}^{(iv)} - 5\mu'''u_1 - 13\mu''u_1 - 11\mu'u_1'' - 3\mu u_1''' + 12\mu''u_1^2 + 48\mu'u_1u_1' + 24\mu u_1'^2 + 24\mu u_1 u_1''\right\} \quad (26)$$

The transformations have been expressed in terms of λ, μ, u_1, and W_1 and so involve only two fields besides the parameters λ and μ. [We have assumed $\tilde{x} = x + \lambda(x)$] so (25) and (26) define a two-field realization of the spin-4 current algebra.

ACKNOWLEDGMENT

One of the authors (S.P.) is grateful to CSIR, Government of India, for a Junior Research Fellowship, which made this work possible.

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